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A Note on Matter Superenergy Tensors

S. Deser
Brandeis University
Department of Physics
Waltham, MA 02454, USA
E-mail: deser@brandeis.edu

Abstract

We consider Bel–Robinson-like higher derivative conserved two-index tensors $H_{\mu\nu}$ in simple matter models, following a recently suggested Maxwell field version. In flat space, we show that they are essentially equivalent to the true stress-tensors. In curved Ricci-flat backgrounds it is possible to redefine $H_{\mu\nu}$ so as to overcome non-commutativity of covariant derivatives, and maintain conservation, but they become model- and dimension- dependent, and generally lose their simple "BR" form.

Historically, the nonexistence of a local stress tensor in generally covariant theories such as Einstein's led to a successful search for the next-best thing, the covariantly conserved but four-index and higher derivative Bel–Robinson (BR) tensor [1], quadratic in curvature. Being traceless in D=4, it has no "$T_{\mu\nu}$-like" 2-index contraction. However, this discovery led ineluctably to an equally successful search [2] for matter analogs of BR, despite the presence of perfectly good $T_{\mu\nu}$ there. These quantities resemble BR in being of higher derivative order and quadratic in the "curvatures" of the corresponding fields. In particular, it has recently been shown [3] that there is a natural 2-index conserved BR-version of the Maxwell tensor. In flat space QFT, operators $H_{\mu\nu}$ whose matrix elements behave like those of the stress-tensor are essentially proportional to it [4]: In momentum space, one expects a conserved symmetric 2-tensor to have the form $f(q^2)T_{\mu\nu}(q)$, up to (trivial) identically conserved terms. By continuity, one might expect some similar property to hold, at least for test fields, *i.e.*, in Ricci-flat spaces. The results we obtain here confirm these expectations, at least for $H_{\mu\nu}$ of simple free field models. Using the simplest – scalar and vector – free field models we will first investigate their $H_{\mu\nu}$ in flat space and immediately verify the expectation that they (and their obvious generalizations) are indeed related to the corresponding $T_{\mu\nu}$ by form factors. In gravitational backgrounds we find that while $H_{\mu\nu}$ can be redefined to survive non-commutation of derivatives at least in Ricci-flat spaces, generically they lose their flat space attributes and become model- and dimension-dependent.

In flat space, the new tensors are respectively

$$H^s_{\mu\nu} \equiv \phi_{\mu\alpha}\phi_\nu{}^{\alpha} - \tfrac{1}{2}\,\eta_{\mu\nu}(\phi_{\alpha\beta}\phi^{\alpha\beta})\ , \quad \phi_{\mu\alpha} \equiv \partial^2_{\alpha\mu}\phi \tag{1}$$

and [2]

$$H^v_{\mu\nu} \equiv F_{\mu\lambda\alpha}F_\nu{}^{\lambda\alpha} - \tfrac{1}{4}\,\eta_{\mu\nu}(F_{\lambda\beta\alpha}F^{\lambda\beta\alpha})\ , \quad F_{\mu\lambda\alpha} \equiv \partial_\alpha\, F_{\mu\lambda} \tag{2}$$

for scalar and Maxwell fields, and exhibit the BR form, in terms of the "curvatures" $\phi_{\mu\nu}$ and $F_{\mu\lambda\alpha}$. [The extra derivatives on the fields are clearly completely transparent to taking divergences, symmetry, etc.] Our main point, however, is that, equally obviously, they are simply related to their corresponding stress tensors, through

$$H^s_{\mu\nu} = \tfrac{1}{2}\,\Box T^s_{\mu\nu} \tag{3}$$

and

$$H^v_{\mu\nu} = \tfrac{1}{2}\,\Box T^{\max}_{\mu\nu}\ . \tag{4}$$

Here and throughout we work entirely on-shell, since that is all that matters; in particular, we used the Maxwell field equation

$$\Box F_{\mu\nu} = 0\ . \tag{5}$$

These examples, then, show that the $H_{\mu\nu}$ are precisely of the (momentum space) form $f(q^2)T_{\mu\nu}(q)$, devoid of independent content.

An excursion into curved space is less directly motivated, as there is no expected simple physical "H–T" connection there. Still, it is mildly interesting that any results at all can be drawn in more general backgrounds. It is immediately clear that not only do the definitions (1,2) of H not imply (3,4) but that in neither form does (covariant) conservation of H follow from that of T. The reason is of course the non-commutativity among covariant derivatives, particularly $[D_\mu, \Box] \neq 0$.

A resolution of this impasse exists, however, for spaces that are Ricci-flat: when $\Box$ is replaced by the Lichnerowicz operator [5],

$$\tilde{H}_{\mu\nu} \equiv (LT)_{\mu\nu} \equiv \Box T_{\mu\nu} + 2R_{\mu\alpha\nu\beta}T^{\alpha\beta}\ , \tag{6}$$

commutativity is restored:

$$D^\mu (LT)_{\mu\nu} = L(D^\mu T_{\mu\nu}) = 0\ . \tag{7}$$

The curvature addition in (6) compensates for $[D^\mu, \Box] \neq 0$. This happens in general for and only for Ricci-flat spaces, as is easily verified (recall that the divergence of the Riemann tensor is the curl of Ricci). We have thus rescued, albeit in restricted geometries,[1] a conserved – redefined – $\tilde{H}_{\mu\nu}$, but at the price of losing the desired BR form (1,2). Can the (necessary) shift from $\Box$ to L be made compatible with (1,2)? The answer is a very qualified "yes", for D=4 Maxwell only [3] and not otherwise. First we see why it does not work for scalars: the only difference between (3) and (1) is that the former contains extra terms $\sim \Box D_\mu \phi$ (of course all derivative ordering must be carefully kept here!). But $[\Box, D_\mu]\phi \sim D_\mu \Box\phi + R_{\mu\alpha}\phi_\alpha$ and involves only the Ricci tensor, so we can drop it. This equality of (3) and (1), however, is also its drawback: since only $\tilde{H} = \Box T + RT$ is conserved, it is not (1) alone, but the sum of (1) with RT that is conserved. The Maxwell model, though only in D=4, circumvents the above difficulty precisely because (4) differs from (2) by terms of the form $F(\Box F)$ and these no longer vanish. Instead,

$$0 = D^\alpha[D_\alpha F_{\mu\nu} + D_\nu F_{\alpha\mu} + D_\mu F_{\mu\alpha}] = \Box F_{\mu\nu} + \Big\{ R^\alpha{}_{\nu\lambda\mu} F_\alpha{}^\lambda - (\nu\mu) \Big\}\ . \tag{8}$$

The ensuing RFF terms in reducing (4) to (2), together with the $RT^{\max}$ part of $LT^{\max}$, cancel each other owing to a specifically D=4 Weyl tensor identity [3]; the (covariantized) $H^v_{\mu\nu}$ of (2), being the reduction of $\tilde{H}^v_{\mu\nu}$, is conserved.

[1] In D=4, $H^{\max}_{\mu\nu}$ is also conserved [3] on combined Einstein–Maxwell shell, *i.e.*, for spaces whose Ricci tensor is proportional to $T^{\max}_{\mu\nu}$, due to the special properties of the Maxwell tensor in D=4. This does not hold in other D, nor even in D=4, for the scalar model.

Finally, a remark on the original, gravitational, BR tensor. In the present context, one might hope to construct it in terms of an underlying gauge-variant stress-tensor. This is impossible for a seemingly accidental reason: the trace of $B_{\mu\nu\lambda\beta}$ vanishes in D=4 and so there is no $H_{\mu\nu}$-like candidate. [The only quadratic on-shell 2-index tensor is the square of Weyl, $C_{\mu\alpha\beta\gamma}C^{\nu\alpha\beta\gamma}$, which is a pure trace in D=4.] More generally, this impossibility is understandable from the known fact [6] that (already in flat space) all $T_{\mu\nu}$ candidates for free linear spin >1 gauge fields vary by a superpotential term, under gauge transformations, one that cannot be removed by any curls or $\Box$ operators: Hence $\Box T_{\mu\nu}$ (or more general operators on $T_{\mu\nu}$) is still conserved but remains gauge-variant, even on shell. The required BR process here is more radical, taking individual "curls" of the factors in T, to reach the required $B \sim RR$ form.

In summary, we have seen that in flat space, ostensibly novel conserved symmetric BR-like extensions of matter $T_{\mu\nu}$ are really equivalent to it. In curved but Ricci-flat backgrounds, conservation of at least the extended $\tilde{H}_{\mu\nu} = (LT)_{\mu\nu}$, can be preserved. Even then (for all but the Maxwell field in D=4), since (LT) is no longer of pure BR form (1) or (2), but acquires various non-minimal additions.

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